ELSEVIER

Topology and its Applications 85 (1998) 165–174

TOPOLOGY AND ITS APPLICATIONS

What is left of CH after you add Cohen reals? ☆

I. Juhász *, L. Soukup, Z. Szentmiklóssy

Magyar Tudomanyos Akademia, Matematikai Kutate Intezete, Realtonda U. 13-15, 1053 Budapest, Hungary

Received 7 November 1996

Abstract

The principle CH* concerning elementary submodels is formulated and is shown to be valid in any generic extension obtained by adding any number of Cohen reals to a ground model satisfying CH.

CH* has interesting topological consequences, e.g.:

(i) Every initially ω_1-compact, countably tight T_3 space is compact.

(ii) Let X be a countably tight compact T_2 space; then (a) if S is G_δ-dense in X then every point of X is the limit of a converging ω_1-sequence from S; (b) if $Y \subset X$ with $\mathrm{s}(Y) \leqslant \omega_1$ then $\mathrm{h}(Y) \leqslant \omega_1$; (c) X contains no complete binary tree of closed sets of height ω_2.

(iii) If X is a compact T_2 space with small diagonal then X is metrizable.

Keywords: Continuum Hypothesis; Cohen reals; Elementary submodels; Initially ω_1-compact; Compact; Countably tight

AMS classification: 54A25; 03E35

1. The principle CH*

Our aim here is to formulate a principle that may be considered as a weakening of CH because it will be shown to hold in any generic extension of a ground model satisfying CH by adding any number of Cohen reals. This principle, that will be called CH*, will concern elementary submodels but will include no mention of forcing.

Let us start with the following trivial observation: If CH holds then for every uncountable regular cardinal λ there are many *countably closed* elementary submodels of size ω_1 of $H(\lambda)$. More precisely, they form a stationary (hence cofinal) subset of $[H(\lambda)]^{\omega_1}$. In fact, the existence of such countably closed elementary submodels is equivalent to CH.

☆ Research supported by OTKA grant no. 16391.

* Corresponding author.

0166-8641/98/$19.00 © 1998 Elsevier Science B.V. All rights reserved.
PII S0166-8641(97)00148-X

What happens to these elementary submodels if Cohen reals are added? We are going to denote by $\mathcal{C}_\kappa$ the natural notion of forcing that adds κ many Cohen reals, i.e., $\mathcal{C}_\kappa = \mathcal{F}n(\kappa, 2)$. The following is well known (cf., e.g., [2] or [7]).

Fact 1.1. *Assume that our ground model V satisfies CH, $\kappa \in M \prec H(\lambda)$, $[M]^\omega \subset M$ and G is $\mathcal{C}_\kappa$-generic over V. Then, in $V[G]$,*

$$N = M[G] \prec H(\lambda)^{V[G]}, \quad \textit{where } M[G] = \{val_G(\tau)\colon \tau \in M\}.$$

Of course, if $\kappa > \omega_1$ then $|M| = \omega_1$ implies $|N| = |M| < \kappa$ hence N can no longer be countably closed. However, it will turn out that N has some properties that are weaker than countable closedness but still strong enough to yield interesting consequences. In order to formulate them we first need a definition.

Definition 1.2. Let N be any set and $A \subset N$. We say that B is an N-cover of A if the following three conditions are satisfied.

(a) $A \subset B \subset N$;
(b) $N \cap [B]^\omega$ is cofinal in $[B]^\omega$;
(c) for every $S \in N \cap [B]^\omega$ we have $|S \cap A| = \omega$.

Clearly, if $[N]^\omega \subset N$ then every set $A \subset N$ is its own N-cover. Hence it is clear that both (i) and (ii) in the following theorem are weakenings of the property of countable closedness.

Theorem 1.3. *If $N = M[G]$ is as in Fact 1.1 then*

(i) *$N \cap [N]^\omega$ is cofinal in $[N]^\omega$;*
(ii) *every set $X \in [N \cap On]^{\omega_1}$ has an uncountable subset $A \in [X]^{\omega_1}$ that has an N-cover.*

Proof. (i) is well known and follows easily from the fact that in a CCC extension every countable set of ordinals is contained in a countable ground model set, moreover from the countable closedness of M.

To show (ii) it will suffice to prove that if $p \models \dot{X} \in [M \cap On]^{\omega_1}$ then some extension q of p forces that some uncountable subset of $\dot{X}$ has an N-cover. To this end we first define distinct ordinals $\alpha_\xi \in M \cap On$ and conditions $p_\xi \in \mathcal{C}_\kappa$ for $\xi \in \omega_1$ such that $p_\xi \leqslant p$ and $p_\xi \models \alpha_\xi \in \dot{X}$. We may assume that $\{p_\xi\colon \xi \in \omega_1\}$ is a Δ-system with root q; of course, then $q \leqslant p$.

Let us set for each $\xi \in \omega_1$

$$r_\xi = p_\xi \cap M,$$

and define the $\mathcal{C}_\kappa$-names $\dot{A}$ and $\dot{B}$ as follows:

$$\dot{A} = \{\langle p_\xi, \alpha_\xi\rangle\colon \xi \in \omega_1\}, \qquad \dot{B} = \{\langle r_\xi, \alpha_\xi\rangle\colon \xi \in \omega_1\}.$$

Clearly, $q \Vdash \dot{A} \in [\dot{X}]^{\omega_1}$ & $\dot{A} \subset \dot{B} \subset N$, hence in view of (i) it will suffice to show that q forces part (c) of Definition 1.2. This, on the other hand is an easy consequence of the fact that every extension of q can only be incompatible with only finitely many of the conditions p_ξ. $\square$

Let us now introduce the following piece of notation.

Definition 1.4. For any regular cardinal λ we denote by $\mathcal{N}_\lambda$ the collection of those elementary submodels $N \prec H(\lambda)$ that satisfy $|N| = \omega_1 \subset N$ as well as (i) and (ii) of Theorem 1.3.

Finally, we are ready to formulate our principle CH*.

Definition 1.5. CH* is the statement that for every large enough regular cardinal λ the collection $\mathcal{N}_\lambda$ is cofinal in $[H(\lambda)]^{\omega_1}$.

Clearly, CH* is a consequence of CH, moreover from Fact 1.1 and Theorem 1.3 we immediately get the following result.

Theorem 1.6. *If $V \models CH$ then for any cardinal κ we have $V^{\mathcal{C}_\kappa} \models CH^*$.*

In fact, what can be shown is that for any regular $\lambda > \kappa$, in $V^{\mathcal{C}_\kappa}$, the collection $\mathcal{N}_\lambda$ is even stationary in $[H(\lambda)]^{\omega_1}$.

2. The Main Lemma

It will turn out that all the results that we deduce from CH* make use of a single lemma concerning initially ω_1-compact and countably tight T_3 spaces belonging to an elementary submodel $N \in \mathcal{N}_\lambda$. Therefore we call this our Main Lemma. Before formulating and proving it, however, we have to give several auxiliary results.

Lemma 2.1. *Let N be an elementary submodel of some $H(\lambda)$ with $\omega_1 \subset N$ and X be a topological space such that $X \in N$. Assume that A, B are subsets of X with $\overline{A} \cap \overline{B} = \emptyset$ and $\widetilde{A}$, $\widetilde{B}$ are N-covers of A and B, respectively. Then $|\widetilde{A} \cap \widetilde{B} \cap X| \geqslant \omega$ implies that there is an ω_2-dyadic system of closed sets in X.*

Proof. By our assumption there is a set $T \subset \widetilde{A} \cap \widetilde{B} \cap X$ with $|T| = \omega$. In view of Definition 1.2(b) there are sets $S_A \in N \cap [\widetilde{A}]^\omega$ and $S_B \in N \cap [\widetilde{B}]^\omega$ with $T \subset S_A \cap S_B$. Then, clearly, $S = S_A \cap S_B \cap X$ is also in N and $|S| = \omega$ as $T \subset S$.

The ω_2-dyadic system will be of the form $\{\overline{S^i_\alpha}\colon \langle \alpha, i\rangle \in \omega_2 \times 2\}$ with $S^i_\alpha \subset S$ for all $\langle \alpha, i\rangle$. It can be defined by a straightforward transfinite recursion using the following claim.

Claim. *If $\mathcal{T} \subset [S]^\omega$ with $|\mathcal{T}| \leqslant \omega_1$ then there are sets $S_0, S_1 \subset S$ with $\overline{S}_0 \cap \overline{S}_1 = \emptyset$ such that $|S_0 \cap T| = |S_1 \cap T| = \omega$ for each $T \in \mathcal{T}$.*

Assume, indirectly, that there is a collection $\mathcal{T}$ contradicting the claim. Since both X and S belong to N, by elementarity, we may assume that $\mathcal{T} \in N$ as well. But then $\omega_1 \subset N$ and $|\mathcal{T}| \leqslant \omega_1$ imply that $\mathcal{T} \subset N$. So for each $T \in \mathcal{T}$ we have $T \in N \cap [\widetilde{A}]^{\omega}$ and $T \in N \cap [\widetilde{B}]^{\omega}$, hence Definition 1.2(c) implies $|T \cap A| = |T \cap B| = \omega$. But then $S_0 = S \cap A$ and $S_1 = S \cap B$ clearly satisfy $\overline{S}_0 \cap \overline{S}_1 = \emptyset$ and $|S_0 \cap T| = |S_1 \cap T| = \omega$ for each $T \in \mathcal{T}$, contradicting the choice of $\mathcal{T}$. This, in turn, completes the proof of the claim and thus of Lemma 2.1. $\square$

From Lemma 2.1 we easily deduce the following lemma.

Lemma 2.2. *Assume $N \in \mathcal{N}_\lambda$ for some regular λ, moreover X is a regular space with $X \in N$ and not containing an ω_2-dyadic system of closed sets. We also assume that the underlying set of X is an ordinal. Then, if $A \in [N \cap X]^{\omega_1}$ and $\widetilde{A} \subset X$ is an N-cover of A, every complete accumulation point of $\widetilde{A}$ in X belongs to $\overline{A}$, the closure of A.*

Proof. Let x be a complete accumulation point of $\widetilde{A}$ and assume that $x \notin \overline{A}$. Then, by the regularity of X, we have an open set U with $x \in U$ and $\overline{U} \cap \overline{A} = \emptyset$.

We also have $|U \cap \widetilde{A}| = \omega_1$, hence applying Theorem 1.3(ii) there is a set $B \in [U \cap \widetilde{A}]^{\omega_1}$ that has an N-cover $\widetilde{B}$. But then we also have $\overline{A} \cap \overline{B} = \emptyset$, moreover $B \subset \widetilde{A} \cap \widetilde{B} \cap X$, hence $|\widetilde{A} \cap \widetilde{B} \cap X| = \omega_1$. Consequently Lemma 2.1 applies and yields an ω_2-dyadic system of closed sets in X, which is impossible. $\square$

Remark. It can be shown that in Lemma 2.2 every complete accumulation point of $\widetilde{A}$ is actually a complete accumulation point of A, however we will not use this stronger result.

The following purely topological lemma is folklore, but we supply its proof because perhaps it has not been published in this generality.

Lemma 2.3. *Assume that X is a topological space with $\mathrm{t}(X) \leqslant F(X) = \omega$ and $\mathcal{F}$ is an ω_1-directed family of closed sets in X. Then for any $A \subset X$ with $\overline{A} \in \mathcal{F}^+$ (i.e., $\overline{A} \cap F \neq \emptyset$ for all $F \in \mathcal{F}$) there is a countable set $S \subset A$ with $\overline{S} \in \mathcal{F}^+$.*

Proof. Assume, indirectly, that $\overline{A} \in \mathcal{F}^+$ but for every countable $S \subset A$ we have $\overline{S} \notin \mathcal{F}^+$. Then by transfinite recursion on $\alpha \in \omega_1$ we define points $p_\alpha \in \overline{A}$, sets $S_\alpha \in [A]^{\leqslant\omega}$ and closed sets $F_\alpha \in \mathcal{F}$ as follows.

If $\alpha \in \omega_1$ and $p_\beta, S_\beta, F_\beta$ have been defined for all $\beta \in \alpha$ then $\overline{\bigcup\{S_\beta\colon \beta \in \alpha\}} \notin \mathcal{F}^+$, hence we can pick $F_\alpha \in \mathcal{F}^+$ such that

$$F_\alpha \cap \overline{\bigcup\{S_\beta\colon \beta \in \alpha\}} = \emptyset \quad \text{and} \quad F_\alpha \subset \bigcap\{F_\beta\colon \beta \in \alpha\}$$

by the ω_1-directedness of $\mathcal{F}$.

Let $p_\alpha \in F_\alpha \cap \overline{A}$ and, using $\mathrm{t}(X) \leqslant \omega$, choose $S_\alpha \in [A]^{\leqslant\omega}$ such that $p_\alpha \in \overline{S}_\alpha$. This completes the recursion.

Now, it is obvious that $\{p_\alpha\colon \alpha \in \omega_1\}$ is a free sequence in X, contradicting $F(X) = \omega$. $\square$

We are now ready to prove our Main Lemma.

Lemma 2.4 (Main Lemma). *Let X be an initially ω_1-compact T_3 space with $\mathrm{t}(X) \leqslant \omega$ and $\mathcal{F}$ be an ω_1-directed family of closed sets in X. We also assume that the underlying set of X is an ordinal. Assume that $N \in \mathcal{N}_\lambda$ and $X, \mathcal{F} \in N$, moreover set*

$$\mathcal{F}_N = \{\overline{F \cap N}\colon F \in \mathcal{F} \cap N\}.$$

Then

$$\bigcap \mathcal{F}_N \subset \bigcap \mathcal{F}.$$

Proof. It is well known that the assumptions on X imply $F(X) = \omega$. Also, there is no ω_1-dyadic system of closed sets in X. Indeed, otherwise we could find a continuous map f of some closed subspace Y of X onto 2^{ω_1}, as, e.g., in [3, p. 70]. Since f preserves initial ω_1-compactness and $\mathrm{w}(2^{\omega_1}) = \omega_1$ it would follow that f is also a closed map. However, as is well-known, closed maps preserve countable tightness, hence we are led to the absurd conclusion that $\mathrm{t}(2^{\omega_1}) = \omega$!

Now let $x \in \bigcap \mathcal{F}_N$; to show that $x \in \bigcap \mathcal{F}$ by the regularity of X it clearly suffices to show that for any open neighbourhood U of x we have $\overline{U} \in \mathcal{F}^+$. To this end let us fix a smaller neighbourhood V of x with $\overline{V} \subset U$.

Using Theorem 1.3(i) and the ω_1-directedness of $\mathcal{F}$ it is easy to see that $\mathcal{F} \cap N$ is also ω_1-directed. Consequently one can easily define a sequence $\{F_\alpha\colon \alpha \in \omega_1\} \subset \mathcal{F} \cap N$ such that $\alpha < \beta$ implies $F_\alpha \supset F_\beta$ and for every $F \in \mathcal{F} \cap N$ there is $\alpha \in \omega_1$ with $F_\alpha \subset F$ (note that $|\mathcal{F} \cap N| \leqslant |N| = \omega_1$).

Now let us pick for each $\alpha \in \omega_1$ a point $x_\alpha \in V \cap F_\alpha \cap N$, this is possible because $x \in \overline{F_\alpha \cap N}$ and $x \in V$. Note that then for every set $a \in [\omega_1]^{\omega_1}$ we have $\{x_\alpha\colon \alpha \in a\} \in \mathcal{F}_N^+$. Thus if there is a point y with $|\{\alpha \in \omega_1\colon x_\alpha = y\}| = \omega_1$, then $\{y\} \in \mathcal{F}_N^+$, hence using $y \in N$ also $\{y\} \in \mathcal{F}^+$. But then $y \in V \subset U$ implies $\overline{U} \in \mathcal{F}^+$ as well.

If, on the other hand, for all y we have $|\{\alpha \in \omega_1\colon x_\alpha = y\}| \leqslant \omega$ then $Y = \{x_\alpha\colon \alpha \in \omega_1\} \in [N \cap X]^{\omega_1}$ hence using Theorem 1.3(ii) there is a set $A \in [Y]^{\omega_1}$ that has an N-cover $\widetilde{A}$. Since $X \in N$ we can clearly assume that $\widetilde{A} \subset X$, because $\widetilde{A} \cap X$ is also an N-cover of A.

Now all the assumptions of Lemma 2.2 are satisfied, hence every complete accumulation point of $\widetilde{A}$ is contained in $\overline{A} \subset \overline{V} \subset U$. Consequently, as X is initially ω_1-compact, we must have $|\widetilde{A} \setminus U| \leqslant \omega$. Let us fix a set $T \in N \cap [\widetilde{A}]^\omega$ such that $\widetilde{A} \setminus U \subset T$.

Then $\widetilde{A} \setminus T \in \mathcal{F}_N^+$, hence applying Lemma 2.3 and property (b) of the N-covers we can find a set $S \in N \cap [\widetilde{A} \setminus T]^\omega$ such that $\overline{S} \in \mathcal{F}_N^+$ as well. But then $S \in N$, hence $\overline{S} \in N$ imply that $\overline{S} \in \mathcal{F}^+$, consequently from $S \subset U$ we also get $\overline{U} \in \mathcal{F}^+$, and thus the proof is completed. $\square$

3. Applications of CH*

In this section we are going to present several interesting consequences of CH*. Since CH* implies CH, it is natural that most of these consequences, though not all, have been known to follow from CH. Many of them were also proven to hold in generic extensions obtained by adding ω_2 Cohen reals to a ground model satisfying CH, especially in [1] that gave the main impetus to our present work.

The first result of this section was proved under CH, independently, by van Dowen and Dow, see, e.g., [1].

Theorem 3.1 (CH*). *Every initially ω_1-compact T_3 space X of countable tightness is compact.*

Proof. We may of course assume that the underlying set of X is an ordinal. Now let $\mathcal{F}$ be any maximal filter of closed sets in X, we shall show that $\bigcap \mathcal{F} \neq \emptyset$.

To this end we apply CH* and choose a large enough regular λ and an elementary submodel $N \in \mathcal{N}_\lambda$ with both $X, \mathcal{F} \in N$. Now, since X is initially ω_1-compact and $\mathcal{F}$ is maximal, $\mathcal{F}$ is clearly even ω_2-directed. Also, if F is a closed set with $F \in \mathcal{F}^+$ then actually $F \in \mathcal{F}$. So using Lemma 2.3 for $\mathcal{F}$ we get that for every $F \in \mathcal{F}$ there is an $S \in [F]^{\leqslant\omega}$ such that $\overline{S} \in \mathcal{F}$ as well. Reflecting this statement down to N we may now conclude that for every $F \in \mathcal{F} \cap N$ there is a set $S \in N \cap [F]^{\leqslant\omega}$ such that $\overline{S} \in \mathcal{F} \cap N$ as well. But from $S \in N$ we have $S \subset N$ as well, so $\overline{S} = \overline{S \cap N} \in \mathcal{F}_N$. Consequently we clearly have $\bigcap(\mathcal{F} \cap N) = \bigcap \mathcal{F}_N$, and here the left hand side is nonempty because $|\mathcal{F} \cap N| \leqslant \omega_1$ and X is initially ω_1-compact. Thus we have by our Main Lemma 2.4 that $\bigcap \mathcal{F} \supset \bigcap \mathcal{F}_n \neq \emptyset$ and this completes the proof that X is compact. $\square$

Our next result seems to be new even under CH. It concerns compact T_2 spaces of countable tightness (in what follows CCT spaces), but of course, in view of Theorem 3.1, it could be formulated for initially ω_1-compact T_3 spaces instead of compact T_2 ones.

Theorem 3.2 (CH*). *Let X be a CCT space and $S \subset X$ be G_δ-dense in X, i.e., $S \cap H \neq \emptyset$ for every nonempty G_δ set $H \subset X$. Then $X = \ell_{\omega_1}(S)$, i.e., every point x of X is the limit of a converging ω_1-sequence of points of S.*

Proof. If $x \in S$ this is trivial, so assume $x \in X \setminus S$. (Of course, we may assume again that $X \subset On$.) Let $\mathcal{F}$ be the family of all those closed G_δ sets in X that contain x.

Using CH* choose a regular λ and an elementary submodel $N \in \mathcal{N}_\lambda$ such that $X, \mathcal{F}, x, S \in N$. Since $\mathcal{F}$ is clearly ω_1-directed, the Main Lemma 2.4 can be applied, hence we have

$$\emptyset \neq \bigcap \mathcal{F}_N \subset \bigcap \mathcal{F} = \{x\}.$$

Consequently, for each $F \in \mathcal{F} \cap N$ we have $x \in \overline{F \cap N}$. In fact, we claim that $x \in \overline{F \cap N \cap S}$ is also true. Indeed, first of all we have $x \in \overline{F \cap S}$ since otherwise we had an open set G with $x \in G$ and $G \cap F \cap S = \emptyset$ which is impossible as $G \cap F$ is a nonempty G_δ set. So by the countable tightness of X there is a countable set $T \subset F \cap S$ with $x \in \overline{T}$ and by elementarity and $x, S, F \in N$ we may assume that $T \in N$ as well. But then

$$x \in \overline{T} \subset \overline{F \cap N \cap S}.$$

Now, similarly as in the proof of Lemma 2.4 we can define a decreasing ω_1-sequence $\{F_\alpha\colon \alpha \in \omega_1\} \subset \mathcal{F} \cap N$ such that for every $F \in \mathcal{F} \cap N$ we have $F_\alpha \subset F$ for some $\alpha \in \omega_1$. But then we clearly have

$$x \in \bigcap \{\overline{F_\alpha \cap N \cap S}\colon \alpha \in \omega_1\} \subset \bigcap \mathcal{F}_N = \{x\},$$

hence if $x_\alpha \in F_\alpha \cap N \cap S$ for each $\alpha \in \omega_1$ then the ω_1-sequence $\{x_\alpha\colon \alpha \in \omega_1\} \subset S$ must converge to x, and the proof is completed. □

The following two statements were established in [1] in $V^{C_{\omega_2}}$ for a ground model V satisfying CH.

Corollary 3.3 (CH*). *If X is CCT and $p \in X$ with $\chi(p, X) > \omega$ then $p \in \ell_{\omega_1}(X \setminus \{p\})$, i.e., p is a nontrivial ω_1-limit in X.*

Proof. This is immediate from Theorem 3.2 because $\chi(p, X) > \omega$ is clearly equivalent to $X \setminus \{p\}$ being G_δ-dense in X. □

Corollary 3.4 (CH*). *If X is CCT and the subspace $Y \subset X$ is pseudocompact, then $\overline{Y} = \ell_{\omega_1}(Y)$.*

Proof. Indeed, this follows from Theorem 3.2 because if Y is pseudocompact then it is G_δ-dense in $\overline{Y}$. □

The following result was a known problem under CH and was proven, under CH, in [6]. As it turns out almost the same proof now yields the same conclusion from CH*.

Theorem 3.5 (CH*). *If a compact T_2 space X has small diagonal then X is metrizable.*

Proof. If $\mathrm{t}(X) > \omega$ held then, by the main result of [6], X would contain a nontrivial convergent ω_1-sequence which contradicts having small diagonal.

So X is CCT and consequently so is the space X_Δ obtained by collapsing the diagonal Δ to a single point in X^2. Indeed, this is so because the CCT property is preserved under products and continuous images. But then we must have $\chi(\Delta, X_\Delta) = \omega$, since otherwise, by Corollary 3.3, $X_\Delta \setminus \{\Delta\}$ would contain an ω_1-sequence converging to Δ, hence, equivalently, $X^2 \setminus \Delta$ would contain such a sequence converging to the diagonal Δ, contradicting that X has small diagonal. However, we clearly have $\chi(\Delta, X^2) = \chi(\Delta, X_\Delta)$, hence X has a G_δ diagonal and so is metrizable. □

The following consequence of CH* is again easily deduced from our Main Lemma. It will, however, play an essential role below.

Theorem 3.6 (CH*). *Assume that X is CCT and $\mathcal{F}$ is an ω_1-directed family of closed sets in X such that $d(F) \leqslant \omega_1$ for each $F \in \mathcal{F}$. Then there is a subfamily $\mathcal{F}' \in [\mathcal{F}]^{\leqslant\omega_1}$ such that $\bigcap \mathcal{F}' = \bigcap \mathcal{F}$.*

Proof. We may assume again that the underlying set of X is an ordinal and from CH* we can find some regular λ and $N \in \mathcal{N}_\lambda$ with both $X, \mathcal{F} \in N$.

Now, for each $F \in \mathcal{F} \cap N$ by elementarity there is a dense subset S of F with $|S| \leqslant \omega_1$ and $S \in N$. From $\omega_1 \subset N$ then we also have $S \subset N$, hence $\overline{F \cap N} = \overline{S} = F$. In other words, we have $\mathcal{F} \cap N = \mathcal{F}_N$, so by the Main Lemma $\mathcal{F}' = \mathcal{F} \cap N$ is as required. □

An immediate consequence of Theorem 3.6 is that in a CCT space X there is no strictly decreasing ω_2-sequence of closed sets of density $\leqslant \omega_1$. This fact is used in the next result.

Theorem 3.7 (CH*). *If X is CCT and $Y \subset X$ satisfies $\mathrm{s}(Y) \leqslant \omega_1$ then $\mathrm{h}(Y) \leqslant \omega_1$ as well.*

Proof. Applying [3, 2.26] just as in the proof of [3, 3.13], we have $\mathrm{z}(Y) = \mathrm{hd}(Y) \leqslant \mathrm{s}(Y) \cdot \mathrm{t}(X)^+ \leqslant \omega_1$ by our assumptions. Assume, indirectly, that $\mathrm{h}(Y) > \omega_1$, then there is a strictly decreasing ω_2-sequence $\{H_\alpha\colon\ \alpha \in \omega_2\}$ of closed subsets of Y. But then the sequence $\{\overline{H_\alpha}\colon\ \alpha \in \omega_2\}$ is also strictly decreasing, moreover we have $d(\overline{H_\alpha}) \leqslant d(H_\alpha) \leqslant \omega_1$, and this is impossible by our above remark. □

Our last consequence of CH* concerns complete binary trees of closed sets in a CCT space X. The proof will use Theorem 3.6 and some ZFC results that seem to be new and interesting in themselves.

A complete binary tree of closed sets of height κ in a space X (or simply a Cantor κ-tree in X) is a collection $\{F_t\colon\ t \in 2^{<\kappa}\}$ where each F_t is a nonempty closed set in X such that $t \supset s$ implies $F_t \subset F_s$ and $F_{t^\frown 0} \cap F_{t^\frown 1} = \emptyset$ for each $t \in 2^{<\kappa}$. We denote by $Tr_\kappa(X)$ the family of all Cantor κ-trees in X. Our aim now is to show that if κ is regular and X is a CCT space in which there is a Cantor κ-tree, then there is also one whose members are all separable. First, however, we prove the following lemma.

Lemma 3.8. *Assume $cf(\kappa) > \omega$, X is a CCT space and $T = \{F_t\colon\ t \in 2^{<\kappa}\} \in Tr_\kappa(X)$. Then there is a countable set $S \subset X$ and an index $t \in 2^{<\kappa}$ such that for each $s \in 2^{<\kappa}$ with $s \supset t$ we have $\overline{S} \cap F_s \neq \emptyset$.*

Proof. Assume, indirectly, that there are no such S and t. We may then define by transfinite recursion on $\alpha \in \omega_1$ points $p_\alpha \in X$ and indices $t_\alpha \in 2^{<\kappa}$ with $p_\alpha \in F_{t_\alpha}$ and $t_{\alpha_1} \subset t_{\alpha_2}$ for $\alpha_1 < \alpha_2$ as follows.

Suppose that $\alpha \in \omega_1$ and $\{p_\beta, t_\beta\colon\ \beta \in \alpha\}$ have already been chosen with these properties. From $cf(\kappa) > \omega$ it follows that $t = \bigcup\{t_\beta\colon\ \beta \in \alpha\} \in 2^{<\kappa}$, hence by the indirect assumption there is an extension $t_\alpha \supset t$ such that $\overline{\{p_\beta\colon\ \beta \in \alpha\}} \cap F_{t_\alpha} = \emptyset$. We then pick $p_\alpha \in F_{t_\alpha}$ arbitrarily.

Clearly, this construction goes through for all $\alpha \in \omega_1$, and it is also obvious that the sequence $\{p_\alpha\colon \alpha \in \omega_1\}$ obtained in this way is free. But this contradicts that X is CCT. $\square$

Note that this proof only used $F(X) = \omega$. In the following result, however, the compactness of X is used essentially.

Theorem 3.9. *If κ is an uncountable regular cardinal and X is a CCT space with $\mathcal{T}r_\kappa(X) \neq \emptyset$ then there is a Cantor κ-tree $\widetilde{T} \in \mathcal{T}r_\kappa(X)$ such that every element of $\widetilde{T}$ is separable.*

Proof. Let $T = \{F_t\colon t \in 2^{<\kappa}\}$ be a Cantor κ-tree in X. We denote by $\mathcal{H}$ the collection of all pairs $\langle H, t\rangle$ where H is a closed set in X, $t \in 2^{<\kappa}$ moreover for any $s \in 2^{<\kappa}$ with $s \supset t$ we have $F_s \cap H \neq \emptyset$. We also consider the following partial order $\prec$ on $\mathcal{H}$:

$$\langle H, t\rangle \prec \langle H', t'\rangle \quad \text{if } H \subset H' \text{ and } t \supset t'.$$

Note that for any $t \in 2^{<\kappa}$ we clearly have $\langle F_t, t\rangle \in \mathcal{H}$.

Let us now list several properties of this partial ordering $\langle \mathcal{H}, \prec\rangle$.

(1) For every $\langle H, t\rangle \in \mathcal{H}$ there are $\langle H_0, t_0\rangle$, $\langle H_1, t_1\rangle \in \mathcal{H}$ such that $H_0 \cup H_1 \subset H$, $H_0 \cap H_1 = \emptyset$ and $t \subset t_0$, $t \subset t_1$.

Indeed, we may simply put $t_i = t^\frown i$ and $H_i = H \cap F_{t_i}$ for $i \in 2$.

(2) If $\nu \in \kappa$ and $\{\langle H_\alpha, t_\alpha\rangle\colon \alpha \in \nu\}$ is a decreasing sequence in $\mathcal{H}$ (i.e., $\alpha \in \beta \in \nu$ imply $\langle H_\beta, t_\beta\rangle \prec \langle H_\alpha, t_\alpha\rangle$) then there is some $\langle H, t\rangle \in \mathcal{H}$ with $\langle H, t\rangle \prec \langle H_\alpha, t_\alpha\rangle$ for all $\alpha \in \nu$. In other words, the poset $\langle \mathcal{H}, \prec\rangle$ is κ-closed.

Indeed, this is immediate from the regularity of κ and the compactness of X; we may just set $H = \bigcap\{H_\alpha\colon \alpha \in \nu\}$ and $t = \bigcup\{t_\alpha\colon \alpha \in \nu\}$.

(3) Finally, for any $\langle H, t\rangle \in \mathcal{H}$ there is a $\langle K, s\rangle \in \mathcal{H}$ with $\langle K, s\rangle \prec \langle H, t\rangle$ such that K is separable.

Indeed, by our definition, $\{F_r \cap H\colon r \supset t\}$ is clearly a Cantor κ-tree in H. But then we may apply Lemma 3.8 which yields a countable set $S \subset H$ and a sequence $s \supset t$ such that $F_r \cap \overline{S} \neq \emptyset$ for each $r \supset s$, hence $\langle \overline{S}, s\rangle$ is as required.

Of course, (3) just says that $\mathcal{K} = \{\langle K, s\rangle \in \mathcal{H}\colon d(K) = \omega\}$ is dense in the poset $\mathcal{H}$.

Now, it is standard to construct using properties (1)–(3) a full binary subtree $\{\langle K_t, s_t\rangle\colon t \in 2^{<\kappa}\}$ of $\mathcal{K}$ such that for each $t \in 2^{<\kappa}$ we have $K_{t^\frown 0} \cap K_{t^\frown 1} = \emptyset$. Consequently, the first "co-ordinates" $\{K_t\colon t \in 2^{<\kappa}\}$ of this tree yield us a Cantor κ-tree in X consisting of separable sets, as was required. $\square$

Since the weight of a separable T_3 space is at most c, it may not contain a well-ordered decreasing sequence of closed sets of length c^+. Consequently we immediately have the following ZFC result from Theorem 3.9.

Corollary 3.10. *If X is a CCT space then $\mathcal{T}r_{c^+}(X) = \emptyset$.*

It should be noted that, as is shown, e.g., by the Alexandrov one-point compactifications of large discrete spaces, there is no upper bound for the lengths of decreasing sequences of closed sets in CCT spaces.

Of course, under CH we have $c^+ = \omega_2$ in Corollary 3.10. Combining Theorems 3.6 with 3.9 we can see that ω_2 works as an upper bound under CH* as well.

Corollary 3.11 (CH*). *For every CCT space X we have $\mathcal{T}r_{\omega_2}(X) = \emptyset$.*

The standard proof of the Čech–Pospišil theorem (see, e.g., [3, 3.16]) yields the following statement: If X is a compact T_2 space such that $\chi(p, X) \geqslant \kappa$ holds for each $p \in X$ then $\mathcal{T}r_\kappa(X) \neq \emptyset$. Hence Corollary 3.11 implies that, under CH*, in every CCT space there is a point of character at most ω_1. It was conjectured in [4] that this consequence may be provable in ZFC. As a matter of fact, it may turn out that even Corollary 3.11 is provable in ZFC!

Also, the above conjecture was shown to hold in $V^{\mathcal{C}_{\omega_1}}$ for *any* ground model V. We do not know if the same can be proved about Corollary 3.11.

We conclude this paper with a result that says that under PFA even ω_1 is a (strict) upper bound for the height of a Cantor tree in a CCT space. In view of our above remarks this constitutes a strengthening of A. Dow's result saying that, under PFA, every CCT space has a point of countable character. The proof is quite similar to the proof of this result of Dow given in [5].

Theorem 3.12 (PFA). *If X is a CCT space then $\mathcal{T}r_{\omega_1}(X) = \emptyset$.*

Proof. Assume, indirectly, that $T \in \mathcal{T}r_{\omega_1}(X)$. Then, if P is the standard countably closed notion of forcing that collapses $|X|$ to ω_1, X is no longer compact in V^P. Indeed, there will be many branches of T with empty intersection. On the other hand, as was shown in the proof of Corollary 3.3 in [5], X remains both countably compact and countably tight in V^P. Hence, by Balogh's lemma (see 3.1 in [5]), there is a proper poset Q that adds a free sequence of length ω_1 to X. But then PFA applied to $P * Q$ implies that X contains such a sequence already in V, a contradiction. □

References

[1] A. Dow, Compact spaces of countable tightness in the Cohen models, in: Set Theory and its Applications (Toronto, 1987), Lecture Notes in Math. 1401 (Springer, Berlin, 1989) 55–67.

[2] A. Dow, F. Tall and B. Weiss, New proofs of the consistency of the normal Moore space conjecture I, Topology Appl. 37 (1) (1990) 33–51.

[3] I. Juhász, Cardinal Functions—Ten Years Later, Math. Centre Tract 123 (CWI, Amsterdam, 1980).

[4] I. Juhász, On the minimum character of points in compact spaces, in: Proceedings 1989 Top. Conf. Pécs, Colloquia Mathematica Societatis János Bolyai 55 (North-Holland, Amsterdam, 1993) 365–371.

[5] I. Juhász, Cardinal functions, in: M. Hušek and J. van Mill, eds., Recent Progress in General Topology, Papers from Prague Toposym, 1991 (North-Holland, Amsterdam, 1992) 419–441.

[6] I. Juhász and Z. Szentmiklóssy, Convergent free sequences in compact spaces, Proc. Amer. Math. Soc. 116 (1992) 1153–1160.

[7] S. Shelah, Proper Forcing (Springer, Berlin, 1982).